I

Final Examination

Answer all questions. Each question carries ten points. You should justify your answer and show all details.

1. Let D be the region bounded by the curves $x = 3y^2, x = 5y^2, x + y = 1$, and $x + y = 2$ in the first quadrant. Evaluate the double integral

$$\iint_D \frac{2x+y}{x^{3/2}}\, dA(x,y) \ .$$

2. Consider the triple integral

$$\iiint_\Omega f(x,y,z)\, dV(x,y,z) \ ,$$

where Ω is the region bounded by $x^2+y^2+z^2 = 1,\ x+y+z = 1,\ x,y,z \geq 0$. Express it as (a) an integral in $dzdydx$ and (b) an integral in polar coordinates $d\rho d\varphi d\theta$.

3. Let D be the region bounded by the curves $y = (x-1)^2 + 2$ and $y = 3$ and let C be the boundary of D oriented in the anticlockwise way. Determine the circulation of the field $\mathbf{F} = y\mathbf{i} + (x^2 + \sin y)\mathbf{j}$ around C.

4. Let Γ be the intersection of the cylinder $x^2 + y^2 = 1$ and the plane $x + 2y + z = 1$ oriented in the anticlockwise direction. Find

$$\oint_\Gamma z\, dx + xy\, dy - 6\, dz \ .$$

5. Evaluate the surface integral

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, d\sigma \ ,$$

where S is the part of $z = x^2 + y^2$ pinched between $z = 1, 4$ with normal pointing out and $\mathbf{F} = 3z\mathbf{i} + 5x\mathbf{j} - 2y\mathbf{k}$.

6. Find the work-done by the force

$$\mathbf{E} = y\cos xy\mathbf{i} + (x\cos xy + \frac{1}{1+z})\mathbf{j} - \frac{y}{(1+z)^2}\mathbf{k}$$

on a person who walks from $A(1,0,0)$ to $B(0,1,5\pi/2)$ along the path $t \mapsto (\cos t, \sin t, t)$.

7. Let Ω be the set bounded by $z = 1, y = 1, y = 3$ and $z = 5 - x^2$ and S its boundary. Find the outward flux of the vector field

$$\mathbf{H}(x,y,z) = (5x + \cos y)\mathbf{i} + (y + \sin xz)\mathbf{j} + \cos y\mathbf{k}$$

across S.

8. Evaluate the improper integral

$$\int_0^\infty e^{-x^2} x^2\, dx.$$

You may use the formula

$$\int_{-\infty}^\infty e^{-x^2}\, dx = \sqrt{\pi}.$$

Final Exam II

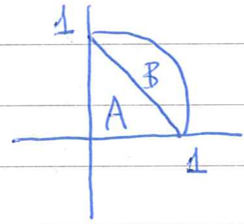
1. Let $u = x/y^2 \in [3,5]$, $v = x+y \in [1,2]$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} -\frac{1}{y^2} & -\frac{2x}{y^3} \\ 1 & 1 \end{vmatrix} = -\frac{y+2x}{y^3}.$$

$$\therefore \iint_D \frac{2x+y}{x^{3/2}}\, dA(x,y) = \int_1^2\int_3^5 \frac{2x+y}{x^{3/2}}\left|-\frac{y^3}{y+2x}\right| dA(u,v)$$

$$= \int_1^2\int_3^5 \frac{y^3}{x^{3/2}}\, dA(u,v) = \int_1^2\int_3^5 \frac{1}{u^{3/2}}\, du\, dv = \cdots \quad \#$$

2. (a) Project to x-y plane, $x, y \geq 0$



Over A, every vertical line hits $z = 1-x-y$ first and then $z = \sqrt{1-x^2-y^2}$. So get

$$\iint_A \int_{1-x-y}^{\sqrt{1-x^2-y^2}} f\, dz\, dA(x,y).$$

Over B, every vertical line up hits $z = \sqrt{1-x^2-y^2}$ but not $1-x-y$. ($z = 1-x-y \leq 0$ in B.) $\therefore$

$$\iint_B \int_0^{\sqrt{1-x^2-y^2}} f\, dz\, dA(x,y).$$

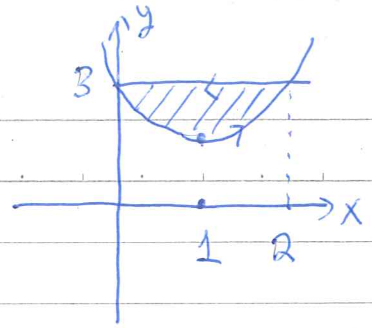
$$\therefore \iiint_\Omega f\, dV = \iint_A \int_{1-x-y}^{\sqrt{1-x^2-y^2}} f\, dz\, dA(x,y) + \iint_B \int_0^{\sqrt{1-x^2-y^2}} f\, dz\, dA(x,y)$$

$$= \int_0^1\int_0^{1-x}\int_{1-x-y}^{\sqrt{1-x^2-y^2}} f\, dz\, dy\, dx + \int_0^1\int_{1-x}^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} f\, dz\, dy\, dx.$$

(b) $$\iiint_\Omega f\, dV = \int_0^{\pi/2}\int_0^{\pi/2}\int_{1/(\sin\varphi\cos\theta + \sin\varphi\sin\theta + \cos\varphi)}^{1} f\rho^2\sin\varphi\, d\rho\, d\varphi\, d\theta.$$

2

3.



$P = y$, $Q = x^2 + \sin y$.

$\dfrac{\partial P}{\partial y} = 1$, $\dfrac{\partial Q}{\partial x} = 2x$.

Green's thm

$$\oint_C P\,dx + Q\,dy = \iint_D (2x-1)\, dA(x,y)$$

$$= \int_0^2 \int_{(x-1)^2+2}^{3} (2x-1)\, dy\, dx = \cdots \quad \#$$

4. Γ projects to the unit circle $x^2+y^2=1$ in the x-y plane. Use θ to parametrize Γ.

$$\Gamma : \theta \mapsto (\cos\theta, \sin\theta, 1-\cos\theta - 2\sin\theta)$$

$$\vec{r}\,'(\theta) = (-\sin\theta, \cos\theta, \sin\theta - 2\cos\theta).$$

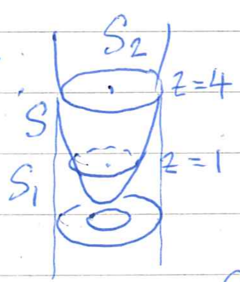
$$\therefore \oint z\,dx + xy\,dy - 6\,dz = \int_0^{2\pi} [(1-\cos\theta-2\sin\theta)(-\sin\theta) + \cos\theta\sin\theta\cos\theta - 6(\sin\theta - 2\cos\theta)]\, d\theta$$

$$= \cdots \quad \#$$

(No way to use Stokes', Green's etc)

direct calculations O.K. but usually got wrong!

5.



Solution. Use Stokes' thm

$$\left(\iint_{S_1} + \iint_{S} + \iint_{S_2}\right) \nabla\times\vec{F}\cdot\hat{n} = 0$$

S_1 the circle $x^2+y^2=1$ at $z=1$, outnormal $= -\hat{k}$

S_2 .... $x^2+y^2=4$ at $z=4$, out normal $= \hat{k}$

$$\nabla\times\vec{F} = -2\hat{i} + 3\hat{j} + 5\hat{k}$$

$$\therefore \iint_{S_1} \nabla\times\vec{F}\cdot\hat{n}\, d\sigma = -5 \times \text{area of } S_1 = -5\pi.$$

$$\iint_{S_2} \nabla\times\vec{F}\cdot\hat{n}\, d\sigma = 5 \times \text{area of } S_2 = 5\pi 2^2 = 20\pi.$$

$$\therefore \iint_S \nabla\times\vec{F}\cdot\hat{n} = -\iint_{S_1} \nabla\times\vec{F}\cdot\hat{n} - \iint_{S_2} \nabla\times\vec{F}\cdot\hat{n}$$

$$= -20\pi + 5\pi = -15\pi \quad \#$$

6. Check $\vec{E}$ is conservative.

Potential $= \sin xy + y/(1+z)$.

$$\therefore \text{ work-done} = \int_A^B \vec{E}\cdot d\vec{r} = \Phi(B) - \Phi(A)$$

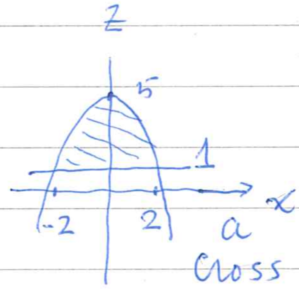
$$= \frac{1}{1+5\pi/2} \,. \#$$

7. Use Divergence Th. $\qquad \operatorname{div}\vec{H} = 6$.

out-flux

$$\iint_S \vec{H}\cdot d\vec{\sigma} = \iiint_\Omega \operatorname{div}\vec{H}\, dV = 6\iiint_\Omega dV.$$

$$= 6\int_1^3 \int_{-2}^{2} \int_1^{5-x^2} dz\, dx\, dy = \cdots \#$$



a cross section of Ω

8. $$\int_0^a e^{-x^2} x^2\, dx = \int_0^a \left(-\frac{1}{2}e^{-x^2}\right)' x\, dx$$

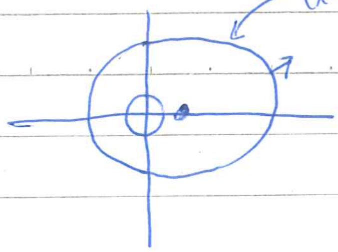
$$= -\frac{1}{2}e^{-x^2}x\Big|_0^a + \frac{1}{2}\int_0^a e^{-x^2}dx$$

$$\to 0 + \frac{1}{2}\int_0^\infty e^{-x^2}dx \quad \text{as } a\to\infty$$

$$\therefore \int_0^\infty e^{-x^2}x^2\, dx = \frac{1}{2}\int_0^\infty e^{-x^2}dx$$

$$= \frac{1}{2}\times\frac{\sqrt{\pi}}{2} = \frac{\sqrt{\pi}}{4} \quad \#$$

9.



Can't apply Green's thm to the region bdd by E. It is because $\vec{A}$ is not bdd at $(0,0)$.

We let C_r be a little circle around $(0,0)$.

By Green's thm

$(x,y)=(r\cos\theta, r\sin\theta)$

$(x',y')=(-r\sin\theta, r\cos\theta)$

$|(x',y')|=r$

$\hat{n}=(\cos\theta,\sin\theta)$

$$\int_E \vec{A}\cdot\hat{n}\,ds = \int_{C_r} \vec{A}\cdot\hat{n}\,ds$$

$$= \int_0^{2\pi}\left(\frac{r\cos\theta}{r^2}\hat{i}+\frac{r\sin\theta}{r^2}\hat{j}\right)\cdot\hat{n}\; r\,d\theta$$

$$= 2\pi.$$

10. (a) If $F_j=\dfrac{\partial\Phi}{\partial x_j}$, then

$$\frac{\partial F_j}{\partial x_i}=\frac{\partial^2\Phi}{\partial x_i\partial x_j}=\frac{\partial^2\Phi}{\partial x_j\partial x_i}=\frac{\partial F_i}{\partial x_j} \quad \#$$

(b)

$$\frac{\partial}{\partial x_j}\Phi(x)=\int_0^1\frac{\partial}{\partial x_j}\left[F_1(t\vec{x})x_1+F_2(t\vec{x})x_2+\cdots+F_n(t\vec{x})x_n\right]dt$$

$$=\int_0^1\left[\frac{\partial F_1}{\partial x_j}(t\vec{x})tx_1+\frac{\partial F_2}{\partial x_j}(t\vec{x})tx_2+\cdots+\frac{\partial F_n}{\partial x_j}(t\vec{x})x_n\right.$$

$$\left.+F_j(t\vec{x})\right]dt$$

$$=\int_0^1\sum_i\left(\frac{\partial F_i}{\partial x_j}(t\vec{x})x_i\right)t+F_j(t\vec{x})\,dt$$

$$=\int_0^1\left(\sum_i\frac{\partial F_j}{\partial x_i}(t\vec{x})x_i\right)t+F_j(t\vec{x})\,dt$$

$$=\int_0^1\frac{d}{dt}\left(F_j(t\vec{x})t\right)dt$$

$$=F_j(t\vec{x})t\Big|_{t=0}^{t=1}=F_j(\vec{x}).$$